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Vicious Random Walkers in the Limit of a Large Number of Walkers

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The vicious random walker problem on a line is studied in the limit of a large number of walkers. The multidimensional integral representing the probability that the p walkers will survive a time t (denoted $P_t^{(p)}$) is shown to be analogous to the partition function of a particular one-component Coulomb gas. By assuming the existence of the thermodynamic limit for the Coulomb gas, one can deduce asymptotic formulas for $P_t^{(p)}$ in the large-p, large-t limit. A straightforward analysis gives rigorous asymptotic formulas for the probability that after a time t the walkers are in their initial configuration (this event is termed a reunion). Consequently, asymptotic formulas for the conditional probability of a reunion, given that all walkers survive, are derived. Also, an asymptotic formula for the conditional probability density that any walker will arrive at a particular point in time t, given that all p walkers survive, is calculated in the limit $t \gg p$.

KEY WORDS: Random walk; Coulomb gas; orthogonal polynomials.

1. INTRODUCTION AND SUMMARY

To commemorate the award of the Boltzmann medal at STATPHYS 15 in Edinburgh, Scotland, M. E. Fisher presented a paper entitled, "Walks, walls, wetting and melting" in a one-hour address—the first lecture of the meeting.[1] Although not present at that occasion, I was in attendance when the paper was presented as a series of lectures by Fisher at the Australian National University during January of the next year (1984). It was during the first of these lectures that the colorfully presented topic of vicious random walkers was introduced, and the probabilistic theory treated in some detail.

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Let us first recall the formulation of the vicious random walker problem on a lattice, according to the lock step model.[1] Consider p random walkers, each an even number of lattice spacings apart, on a one-dimensional lattice. At regular time intervals, each walker must take either a step one lattice site to the left or a step one lattice site to the right, with equal probability 1/2. The walkers are termed vicious, since if any two should arrive at the same site, exchange of gunfire results in the deaths of both walkers.

A basic problem, which arose in the context of the domain wall theory of two-dimensional phase transitions,[2] is to calculate the probability density that all p walkers will survive a large number of steps n and arrive at a particular set of lattice points. This problem was solved by Huse and Fisher.[2] Instead of considering the above lock step model, they studied the corresponding Brownian motion model, which can be thought of as the limit in which the lattice spacings and time increments are taken to zero.

It is the Brownian motion model of vicious random walkers which we will study here. We suppose that initially there are p vicious walkers equally spaced a distance a from each other and distributed at and to the right of the origin. Remarkably, the probability density $Q_t^{(p)}$ that all walkers survive a time t and arrive at the points

$$x_1 < x_2 < \cdots < x_p \tag{1.1}$$

can be calculated exactly for any p.[2] One finds [Ref. 1, Eq. (4.7) with n replaced by Dt and $\sigma = 0$]

$$Q_t^{(p)}(x_1,\dots, x_p) = \frac{\exp[-\sum_{j=1}^{p} (x_j)^2/2Dt - a^2 s_p/Dt]}{(2\pi\, Dt)^{p/2}} \times \prod_{1 \leqslant j < k \leqslant p} \left[\exp\left(\frac{ax_k}{Dt}\right) - \exp\left(\frac{ax_j}{Dt}\right)\right] \tag{1.2}$$

where

$$s_p = \frac{1}{12}\, p(p-1)(2p-1) \tag{1.3}$$

and D is the diffusion constant, defined so that the mean square displacement of a single walker is Dt. Our primary concern will be with the problem, left open by Fisher,[1] of calculating the asymptotic behavior of probabilistic quantities from (1.2) in the limit when both p and t become large.

We will begin our study in Section 2 by providing a derivation of (1.2) from the p-dimensional heat equation. Next, the integral representing the probability that all p walkers survive a time t (denoted $P_t^{(p)}$) is transformed to display an analogy with the partition function of a particular Coulomb gas. This is essential in determining the large-p, large-t behavior of $P_t^{(p)}$, since it is possible to conjecture the functional form of the leading-order behavior of the partition function of the Coulomb gas from physical considerations, which is a vital step in our method of calculation.

In Section 3, the results obtained from the Coulomb gas analogy are used to specify the asymptotic behavior of $P_t^{(p)}$ [see (3.9)–(3.11)]. Further, asymptotic formulas for the probability of a reunion $R_t^{(p)}$ (the probability that the final configuration of walkers is equispaced with spacing a) are specified by (3.17)–(3.19). Combining the results for $P_t^{(p)}$ and $R_t^{(p)}$, the asymptotic behavior of the conditional probability of a reunion, given that all walkers survive, is given by (3.21)–(3.24).

In the final section, the asymptotic behavior of the conditional probability density that a walker will arrive at the point x, given that all walkers survive, is calculated exactly for $t \gg p$. The analysis here relies on a result from the theory of random matrices.[3]

2. THE COULOMB GAS ANALOGUE

2.1. Solution of the p-Dimensional Heat Equation

It was observed by Huse and Fisher[2] that p vicious random walkers on a line can be considered as a single walker in p dimensions, walking in the simplex region (1.1). It follows immediately (although it is not stated explicitly in either ref. 1 or ref. 2) that the probability of survival $Q_t^{(p)}$ must satisfy the p-dimensional heat equation

$$\sum_{j=1}^{p} \frac{\partial^2}{\partial (x_j)^2} Q_t^{(p)} = \frac{2}{D} \frac{\partial}{\partial t} Q_t^{(p)} \tag{2.1}$$

The boundary condition to be satisfied is

$$Q_t^{(p)} = 0 \qquad \text{whenever} \quad x_j = x_{j'}, \quad j, j' = 1, 2, \dots, p \quad (j \neq j) \tag{2.2}$$

while the initial condition is

$$Q_t^{(p)} \underset{t \to 0}{\simeq} \prod_{j=1}^{p} \phi_t^{(p)}(j, x_j) \tag{2.3}$$

where

$$\phi_t^{(p)}(j, x_j) = \left(\frac{1}{2\pi Dt}\right)^{p/2} e^{-[x_j - (j-1)a]^2/2Dt} \tag{2.4}$$

The conditions (2.3) and (2.4) say that for small times the vicious random walkers behave as unrestricted Brownian particles, initially equally spaced (spacing a) and distributed at and to the right of the origin. As a final restriction, we require that other than at the points (2.2), $Q_t^{(p)}$ must be strictly positive.

Since the solution of (2.1) subject to the conditions in the above paragraph is unique, finding the solution derives the formula for $Q_t^{(p)}$. In fact, the solution follows immediately from antisymmetrizing the free solution (2.3) (this procedure is of course standard in the theory of non-interacting fermions in an external field). Thus,

$$Q_t^{(p)} = \det[\phi_t^{(p)}(k, x_j)]_{j,k=1,2,\ldots,p} \tag{2.5}$$

as all of (2.1)–(2.3) are satisfied [(2.3) follows since we have the restriction (1.1)]. The positivity condition follows immediately from the formula (1.2), which in turn follows from straightforward manipulation of the determinant in (2.6) and the van der Monde expansion

$$\det[y_j^{k-1}]_{j,k=1,\ldots,p} = \prod_{1 \leqslant j < k \leqslant p} (y_k - y_j) \tag{2.6}$$

2.2. Probability of Survival and a Coulomb Gas Analogue

Let us now consider the problem of calculating the probability that all p walkers survive, which is obtained from (1.2) by integrating over the simplex region (1.1). From the symmetry of the integrand, the region of integration can be taken as all of $\mathbf{R}^p$ provided we take the absolute value of the product in (1.2) and divide by $p!$.

Denoting this probability by $P_t^{(p)}$, after straightforward manipulation we therefore have

$$P_t^{(p)} = \frac{e^{-a^2 p(p^2-1)/24Dt}}{(2\pi\tau_{p,t})^{p/2}} 2^{p(p-1)/2} I(1) \tag{2.7}$$

where

$$\begin{aligned} I(\gamma) = \frac{1}{p!} \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_p \exp\left(-\frac{\gamma}{2} \sum_{j=1}^{p} \frac{(x_j)^2}{\tau_{p,t}}\right) \\ \times \left| \prod_{1 \leqslant j < k \leqslant p} \sinh \frac{a(x_k - x_j)}{2(Dt\,\tau_{p,t})^{1/2}} \right|^{\gamma} \end{aligned} \tag{2.8}$$

The auxiliary parameter γ and the function $\tau_{p,t}$ [whose form will be chosen for analytical convenience in (2.22) and (2.24)] have been introduced, because then (2.8) can be interpreted as the partition function of a Coulomb gas system.

To see this, we know[4,5] that

$$\phi(x, y) = -\log |\sinh\{\pi(x + iy)/L\}| \tag{2.9}$$

satisfies the two-dimensional Poisson equation

$$\nabla^2\phi(x, y) = -2\pi\delta(x)\,\delta(y) \tag{2.10}$$

subject to the semiperiodic boundary condition $\phi(x, y) = \phi(x, y + L)$. Thus, for a system of p positive two-dimensional charges each of magnitude q, in periodic boundary conditions in the y direction with $L = 2\pi(Dt\,\tau_{p,t})^{1/2}/a$, and confined to a line in the x direction, the Boltzmann factor is

$$\left| \prod_{1 \leqslant j < k \leqslant p} \sinh \frac{a(x_k - x_j)}{2(Dt\,\tau_{p,t})^{1/2}} \right|^{\gamma} \tag{2.11}$$

where x_k denotes the coordinate of the kth particle, and

$$\gamma = q^2/k_{\mathrm{B}} T \tag{2.12}$$

(see upper part of Fig. 1).

Our ability to determine the large-p behavior of (2.7) is dependent on (2.8) representing the partition function of a stable Coulomb gas (up to multiplicative constant terms). We expect a one-component log-potential Coulomb gas to be stable whenever the system (i) is globally charge neutral, (ii) has a finite, nonzero particle density in the thermodynamic limit, and (iii) the potential tends to

$$-\log r \tag{2.13}$$

where r is the particle separation, in the thermodynamic limit.

For condition (i), we introduce a neutralizing charge density $-q\sigma(x)$ with the property

$$\int_{-\infty}^{\infty} \sigma(x)\,dx = p \tag{2.14}$$

This background will couple to the particles to give a further term to the Boltzmann factor. To obtain the prefactor

$$\exp\left[(-\gamma/2) \sum_{j=1}^{p} (x_j)/\tau_{p,t} \right] \tag{2.15}$$

of (2.11) in the integrand in (2.8), we require

$$\int_{-\infty}^{\infty} \sigma(s) \log \left| \sinh \frac{a(x-s)}{2(Dt\,\tau_{p,t})^{1/2}} \right| = \frac{x^2}{2\tau_{p,t}} + c \tag{2.16}$$

For $|x|$ large

$$\sinh \frac{a(x-s)}{2(Dt\,\tau_{p,t})^{1/2}} \sim \frac{a}{2(Dt\,\tau_{p,t})^{1/2}} |x-s| \tag{2.17}$$

which implies that in this limit the lhs of (2.16) is of order $|x|$. Thus, (2.16) cannot be valid for large $|x|$. This means that the domain of the Coulomb gas must be a finite interval.

The dependence of $\tau_{p,t}$ on p and t is obtained from condition (ii) above. First note that since one-component Coulomb systems are conductors, the particle density will exactly equal the background density, provided the latter is slowly varying with respect to the mean spacing between charges. (Were this not the case, there would be a macroscopic electric field within the conductor, so it would not be in equilibrium.) Thus, perhaps the simplest way to ensure condition (ii) is to require that $\sigma(s)$ tends to a positive constant η as p becomes large, for then the above argument implies that the particle density also has the constant value η.

Considering (2.14) and the second sentence after (2.17), for p large we thus have

$$\sigma(x) \sim \begin{cases} \eta, & |x| \leqslant p/2\eta \\ 0, & |x| > p/2\eta \end{cases} \tag{2.18}$$

Substituting (2.18) in (2.16) gives that to leading order, for p large and x fixed,

$$\int_{-p/2\eta}^{p/2\eta} \log \left| \sinh \frac{a(x-s)}{2(Dt\,\tau_{p,t})^{1/2}} \right| ds = \frac{x^2}{2\eta\tau_{p,t}} + c' \tag{2.19}$$

By breaking the range of integration into two intervals $[x, p/2\eta]$ and $[-p/2\eta, x]$, it is straightforward to show that the x-dependent portion of the lhs of (2.19) behaves as

$$\frac{ax^2}{2(Dt\,\tau_{p,t})^{1/2}} \left(1 + \frac{2\exp[-ap/2\eta(Dt\,\tau_{p,t})^{1/2}]}{1-\exp[-ap/2\eta(Dt\,\tau_{p,t})^{1/2}]} \right) \tag{2.20}$$

in the desired limit.

For $p/(Dt\,\tau_{p,t})^{1/2} \to 0$, the final factor in (2.20) behaves as $4\eta(Dt\,\tau_{p,t})^{1/2}/ap$ and thus (2.20) behaves as

$$2x^2\eta/p \tag{2.21}$$

Comparing (2.21) to the x-dependent term on the rhs shows that we require

$$\tau_{p,t} = p/4\eta^2, \qquad p \leqslant Dt \tag{2.22}$$

On the other hand, for $p/(Dt\,\tau_{p,t})^{1/2} \to \infty$, the final factor in (2.20) tends to one and thus the x-dependent portion of the lhs of (2.19) behaves as

$$\frac{ax^2}{2(Dt\,\tau_{p,t})^{1/2}} \tag{2.23}$$

which implies the choice

$$\tau_{p,t} = Dt/(a\eta)^2, \qquad p > Dt \tag{2.24}$$

Together with (2.22), (2.24) defines $\tau_{p,t}$ for large p and t, so that the condition (ii) above is satisfied. It may seen a little disturbing at this stage that the arbitrary positive constant η appears in both (2.22) and (2.24). However, it will transpire that the only way $\tau_{p,t}$ enters our asymptotic formulas is as the argument of a logarithm, so that η is irrelevant to the leading behavior.

The third and final criteria for stability holds if we take the pair potential to be

$$-\log\left\{\left|\sinh\frac{a(x_k - x_j)}{2(Dt\,\tau_{p,t})^{1/2}}\right|\frac{2(Dt\,\tau_{p,t})^{1/2}}{a}\right\} \tag{2.25}$$

This is the original potential, plus a term independent of the particle coordinate, and so will give the Boltzmann factor (2.11) now multiplied by a constant.

Hence the integrand in (2.8) is identical to the Boltzmann factor of a stable Coulomb gas if we multiply the former by

$$e^{\gamma f(p,t)} \tag{2.26}$$

(the Boltzmann factor of the constant terms, that is, those terms independent of the particle coordinates, in the Coulomb gas Hamiltonian). However, the partition function of the Coulomb gas should be an integral

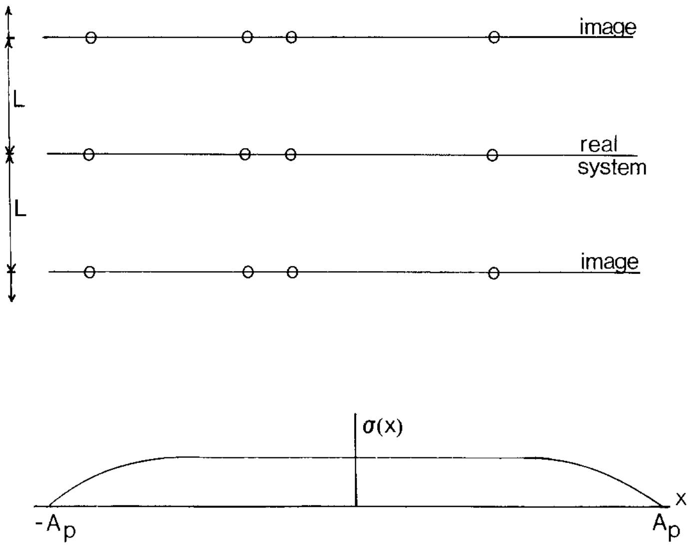

Fig. 1. The Coulomb gas analogue for the probability of survival of p vicious random walkers for a time t. The Coulomb gas consists of p like charges on a line interacting via the logarithmic potential at the reduced temperature $q^2/k_B T = 1$. There are periodic boundary conditions in the y direction with period $L = 2\pi(Dt\,\tau_{p,t})^{1/2}/a$, which can be represented by periodic images of the real system repeated indefinitely. Further, there is a neutralizing background in the range $[-A_p, A_p]$, $A_p = p\eta/2$, responsible for the harmonic attractions toward the origin and the stability of the Coulomb gas.

over the range $x_j \in [-p/2\eta, p/2\eta]$, $j = 1,..., p$. Extending the range of integration to all of $\mathbf{R}^p$, we obtain (2.8) multiplied by (2.26). Due to the Gaussians (2.15) in the integrand, this approximation will not change the value of the free energy. The Coulomb gas analogy is summarized in Fig. 1.

2.3. Conjectured Asymptotic Behavior of the Coulomb Gas Partition Function

Above we argued that for some unspecified $f(p, t)$

$$e^{\gamma f(p,t)} I(\gamma) \tag{2.27}$$

is equal to the partition function of a stable Coulomb gas. Mathematically, stability means

$$\lim_{p \to \infty} \{\gamma f(p, t)/p + (1/p) \log I(\gamma)\} \tag{2.28}$$

exists. Thus, if

$$f(p, t)/p \underset{p \to \infty}{\sim} f_1(p, t) \to \pm\infty \tag{2.29}$$

and the value of the limit (2.28) is

$$g(t, \gamma) \tag{2.30}$$

then

$$I(\gamma) \underset{p \to \infty}{\sim} e^{-\gamma p f_1(p,t) + p g(\gamma, t)} \tag{2.31}$$

The key feature for our purposes is that the γ dependence of the leading behavior is linear in the exponent. If the leading behavior can be computed for any value of $\gamma > 0$, it is therefore known for all $\gamma > 0$. We will use this conjecture to compute the leading behavior of $I(1)$, by first computing exactly $I(2)$.

But before doing this, let us exhibit the correctness of the analogue of (2.31) for the integral

$$J(\gamma) = \frac{1}{p!} \prod_{l=1}^{p} \int_{-\infty}^{\infty} dx_l \left[\exp\left(-\frac{\gamma}{2} \sum_{j=1}^{p} \frac{(x_j)^2}{p} \right) \right] \times \prod_{1 \leqslant j < k \leqslant p} |x_j - x_k|^\gamma \tag{2.32}$$

It is straightforward to check that the three criteria of the above subsection are satisfied, so that (2.32) represents, up to a constant term, the partition function of a stable Coulomb gas. In this case we have the exact evaluation for all γ,[3,6]

$$J(\gamma) = \frac{p^{p/2 + \gamma p(p-1)/2}}{p!} \left(\frac{2\pi}{\gamma} \right)^{p/2} \gamma^{-p(p-1)\gamma/4} \prod_{l=1}^{p} \frac{(\gamma l/2)!}{(\gamma/2)!} \tag{2.33}$$

For p large, Stirling's formula then gives

$$J(\gamma) \sim \exp\left\{ \gamma \left[\frac{1}{2} \sum_{j=1}^{p} j \log j - \frac{p^2(1 + \log 2)}{4} \right] + p h(\gamma) \right\} \tag{2.34}$$

where h is a bounded function, in agreement with the structure of (2.31) and (2.29).

2.4. Exact Evaluation of a Multidimensional Integral

Now we take up the task of evaluating (2.8) with $\gamma = 2$. For this purpose we require the Stieltjes–Wigert polynomials[7] $s_n(x; k)$, which have the orthonormality property

$$\int_0^\infty w(x)\, s_n(x; k)\, s_m(x; k)\, dx = \delta_{n,m} \tag{2.35}$$

$$\text{where } w(x) = \pi^{-1/2} k e^{-k^2 \log^2 x} \tag{2.36}$$

These polynomials are explicitly given by

$$s_n(x; k) = \frac{(-1)^n q^{n/2+1/4}}{\{(1-q)(1-q^2)\cdots(1-q^n)\}^{1/2}} \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\nu^2}(-q^{1/2}x)^\nu \tag{2.37}$$

where

$$q = e^{-1/(2k^2)} \tag{2.38}$$

and $[{n \atop \nu}]$ denotes the Gaussian polynomial. The only property of the Gaussian polynomial we shall require is

$$\begin{bmatrix} n \\ n \end{bmatrix} = 1 \tag{2.39}$$

To make use of these orthonormal polynomials, we first make some simple manipulations to the integrand in (2.8) to obtain the representation

$$\begin{aligned} I(2) = {} & \frac{2^{-p(p-1)}}{p!} \exp\left(\frac{-a^2p^3}{4Dt}\right) \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_p \\ & \times \exp\left\{-\sum_{j=1}^{p}\left[\frac{(x_j)^2}{\tau_{p,t}} + \frac{ax_j}{(Dt\,\tau_{p,t})^{1/2}}\right]\right\} \\ & \times \prod_{1\leqslant j<k\leqslant p}\left[\exp\frac{ax_k}{(Dt\,\tau_{p,t})^{1/2}} - \exp\frac{ax_j}{(Dt\,\tau_{p,t})^{1/2}}\right] \end{aligned} \tag{2.40}$$

The change of variables

$$x_j = [(Dt\,\tau_{p,t})^{1/2}/a] \log y_j, \qquad j = 1, 2,..., p \tag{2.41}$$

then gives

$$\begin{aligned} I(2) = {} & \frac{2^{-p(p-1)}}{p!} (\pi\tau_{p,t})^{p/2} e^{-a^2p^3/4Dt} \prod_{l=1}^{p}\left[\int_0^\infty dy_l\, w(y_l)\right] \\ & \times \prod_{1\leqslant j<k\leqslant p} (y_k - y_j)^2 \end{aligned} \tag{2.42}$$

where $w(x)$ is given by (2.35) with

$$k = (Dt)^{1/2}/a \tag{2.43}$$

The multidimensional integral in (2.42) can now be evaluated using the Stieltjes–Wigert polynomials (2.37) in accordance with the following general result.

Theorem 2.1. Let $\{s_n(x)\}_{n=0,1,...}$ be a set of orthonormal polynomials with respect to a given weight function $w(x)$ on the interval $[a, b]$ (which may be semi-infinite or infinite). Suppose

$$s_n(x) = a_n x^n + \cdots \tag{2.44}$$

Then

$$\frac{1}{p!} \prod_{l=1}^{p} \int_a^b dx_l\, w(x_l) \prod_{1 \leqslant j < k \leqslant p} (x_k - x_j)^2 = \left(\prod_{l=0}^{p-1} \alpha_l \right)^{-2} \tag{2.45}$$

The proof of this result is straightforward. See, for example, ref. 3, Section 17.3.

From (2.37)–(2.39) and (2.43), for the Stieltjes–Wigart polynomials relevant to the evaluation of (2.42),

$$s_n(x; k) = q^{(n+1/2)^2} \{(1-q)(1-q^2) \cdots (1-q^n)\}^{-1/2} x^n + \cdots \tag{2.46}$$

where

$$q = e^{-a^2/2Dt} \tag{2.47}$$

Hence, according to Theorem 2.1, we have the exact result

$$I(2) = 2^{-p(p-1)} (\pi \tau_{p,t})^{p/2}\, e^{a^2(p^3 - p)/12Dt} \prod_{k=1}^{p-1} (1-q^k)^{p-k} \tag{2.48}$$

3. ASYMPTOTIC FORMULAS

3.1. The Probability of Survival for Many Walkers

From the conjectured behavior (2.31) we have that to leading order

$$I(1) \underset{p \to \infty}{\sim} [I(2)]^{1/2} \tag{3.1}$$

The asymptotic behavior of $I(2)$ is determined by the product in (2.48), which we shall denote by $S(p, t)$.

We are interested in the large-p and -t behavior of $S(p, t)$, which depends on the behavior of the ratio p/t. To obtain the expansions, it is useful to first take the logarithm of $S(p, t)$. For $p/t \to 0$, use of the expansion

$$1 - q^k \sim \frac{ka^2}{2Dt} \left(1 - \frac{ka^2}{4Dt}\right) \tag{3.2}$$

gives

$$\log S(p,t) \sim -\frac{1}{2} p(p-1) \log \frac{2Dt}{a^2} + \sum_{k=1}^{p-1} \log k! - \frac{a^2 p^3}{24Dt} \tag{3.3}$$

In the case $p/t \to \text{const}$, the summation obtained from taking the logarithm has the functional form

$$\sum_{k=1}^{p-1} f(k/p) \tag{3.4}$$

where $f(x)$ is a well-behaved function. This is a Riemann sum and yields the leading behavior

$$\log S(p,t) \sim p^2 \int_0^1 (1-s) \log(1 - e^{-pa^2 s/2Dt})\, ds \tag{3.5}$$

For $p/t \to \infty$, the functional form is the Riemann sum

$$\sum_{k=1}^{p-1} g(k/t) \tag{3.6}$$

which, together with the result

$$\int_0^\infty \log(1 - e^{-t})\, dt = -\pi^2/6 \tag{3.7}$$

gives

$$\log S(p,t) \sim -\pi^2 p\, Dt/3a^2 \tag{3.8}$$

Combining (2.48) and (3.3), (3.5), and (3.8), we thus have the asymptotic behavior of $I(2)$ for large p, and consequently from (3.1) the behavior of $I(1)$. Substituting the latter results in (2.7), we thus have that for large p, the probability that all walkers survive exhibits the behavior

$$P_t^{(p)} \underset{p/t \to 0}{\sim} \left(\frac{2Dt}{a^2}\right)^{-p(p-1)/4} \exp\left(-\frac{p}{2} \log p + \frac{1}{2} \sum_{k=1}^{p-1} \log k! - \frac{a^2 p^3}{48Dt}\right) \tag{3.9}$$

$$P_t^{(p)} \underset{p/t \to \text{const}}{\sim} t^{-p/2} \exp\left[\frac{p^2}{2} \int_0^1 (1-s) \log\left(1 - \exp\frac{-pa^2 s}{2Dt}\right) ds\right] \tag{3.10}$$

$$P_t^{(p)} \underset{p/t \to \infty}{\sim} t^{-p/2} \exp\left(\frac{-\pi^2 p\, Dt}{3a^2}\right) \tag{3.11}$$

From our argument suggesting (3.1), the multiplicative factors not specified in these formulas [$h(t, p)$, say] each have the property that

$$\lim_{t, p \to \infty} \frac{1}{p} \log h(t, p) \tag{3.12}$$

is bounded. Terms with this property in (2.48) [which includes the arbitrary constant η in the definition of $\tau_{p,t}$ (2.22) and (2.24)] therefore have not been included in the formulas (3.9)–(3.11).

For p fixed, Fisher [ref. 1, Eqs. (4.9) and (4.10) with n replaced by Dt and $b = 1$] obtained the large-t expansion

$$P_t^{(p)} \sim p^{-p^2/2}(2\pi)^{-p/2} J(1)(Dt/a^2)^{-p(p-1)/4} [1 + O(p^3 a^2/Dt)] \tag{3.13}$$

where $J(1)$ is given by (2.32). Using the exact evaluation of $J(1)$, (2.33), and expanding for large p using Stirling's formula reclaims the leading behavior of (3.9) up to terms with the property stated in the sentence involving (3.12) above.

3.2. The Probability of a Reunion

In the application of the vicious random walker problem to the domain wall theory of melting given by Huse and Fisher,[2] it is necessary to calculate the asymptotic behavior of the probability density of the final configuration being identical to the initial configuration (up to a translation). Thus, in (1.2) we require

$$x_j - x_k = (j-k)a \tag{3.14}$$

The precise location of each walker is then specified by specifying the mean

$$\bar{x} = \sum_{j=1}^{p} x_j/p \tag{3.15}$$

Let us denote the probability density of a reunion of p walkers at time t for a particular $\bar{x}$ by $r_t^{(p)}(\bar{x})$. Then, as noted by Fisher,[1] from (1.2) we can immediately conclude

$$r_t^{(p)}(\bar{x}) = \frac{e^{-p(\bar{x}-\bar{x}_0)^2/2Dt}}{(2\pi\, Dt)^{p/2}} \prod_{k=1}^{p-1} (1 - e^{-a^2k/Dt})^{p-k} \tag{3.16}$$

where $\bar{x}_0$ is defined by (3.15) with $x_j = (j-1)a$.

Recalling the definition of $S(p, t)$ given between (3.1) and (3.2), we see that the product in (3.16) is exactly $S(p, t/2)$. Thus, the asymptotic behavior is given by (3.3), (3.5), and (3.8).

Integrating $r_t^{(p)}(\bar{x})$ over all $\bar{x}$, we therefore obtain that the probability of a reunion anywhere, to be denoted $R_t^{(p)}$, behaves for large p as

$$R_t^{(p)} \underset{p/t \to 0}{\sim} A_p t^{-(p^2-1)/2} \exp\left(\sum_{k=1}^{p-1} \log k! - \frac{a^2 p^3}{12Dt}\right) \tag{3.17}$$

$$R_t^{(p)} \underset{p/t \to \text{const}}{\sim} p^{-1/2}(2\pi\, Dt)^{-(p-1)/2} \times \exp\left[p^2 \int_0^1 (1-s)\log\left(1 - \exp\frac{-pa^2 s}{Dt}\right) ds\right] \tag{3.18}$$

$$R_t^{(p)} \underset{p/t \to \infty}{\sim} p^{-1/2}(2\pi\, Dt)^{-(p-1)/2} \exp\left(\frac{-\pi^2 p\, Dt}{6a^2}\right) \tag{3.19}$$

where

$$A_p = p^{-1/2}(2\pi D)^{-(p-1)/2}\,(D/a^2)^{-p(p-1)/2} \tag{3.20}$$

Unlike (3.9)–(3.11), these results are completely rigorous, and further terms can be specified if desired. Equation (3.17) is consistent with Eqs. (4.16) and (4.17) of Fisher,[1] in which an asymptotic formula for $R_t^{(p)}$ is given for large t and p fixed.

3.3. The Conditional Probability of a Reunion

The conditional probability of a reunion, given that all walkers survive ($S_t^{(p)}$, say) is simply

$$S_t^{(p)} = R_t^{(p)}/P_t^{(p)}$$

Hence from (3.9)–(3.11) and (3.17)–(3.19) we have the large-p, large-t expansions

$$S_t^{(p)} \underset{p/t \to 0}{\sim} C_p t^{-(p-1)(p+2)/4} \exp\left(\frac{p}{2}\log p + \frac{1}{2}\sum_{k=1}^{p-1} \log k! - \frac{a^2 p^3}{16Dt}\right) \tag{3.21}$$

$$S_t^{(p)} \underset{p/t \to \text{const}}{\sim} \exp\left\{p^2 \int_0^1 (1-s)\log\left[\frac{1-e^{-pa^2 s/Dt}}{(1-e^{-pa^2 s/2Dt})^{1/2}}\right] ds\right\} \tag{3.22}$$

where

$$C_p = (D/2a^2)^{-p^2/4} \tag{3.23}$$

For large p and $p/t \to \infty$, from (3.11) and (3.19), the terms $R_t^{(p)}$ and $P_t^{(p)}$ are identical. From the comment below (3.11), this means

$$S_t^{(p)} \underset{p/t \to \infty}{\sim} O(e^{-ph(p,t)}) \tag{3.24}$$

where

$$h(p, t) \underset{p/t \to \infty}{\sim} O(1) \tag{3.25}$$

The exponent $(p-1)(p+2)/4$ in (3.21) is that obtained by Fisher [ref. 1, Eq. (4.5)] for the asymptotic expansion of $S_t^{(p)}$ for t large and p fixed.

4. DENSITY PROFILE FOR SURVIVING WALKERS

The final problem to be addressed is to calculate the asymptotic behavior of the conditional probability density [$\rho_t^{(p)}(x)$, say] that any walker will arrive at a particular point in time t, given that all walkers survive. From (1.2), this is given by

$$\rho_t^{(p)}(x) = \frac{\delta}{\delta A(x)} \prod_{l=1}^{p} \int_R dx_l \, [1 + A(x_l)] \, |Q_t^{(p)}(x_1, ..., x_p)| \, |_{A=0} / P_t^{(p)} \tag{4.1}$$

where $\delta/\delta A$ denotes functional integration and R denotes the region (1.1). Since the integrand is symmetric in each variable, the simplex region (1.1) can be replaced by all of $\mathbf{R}^p$ provided we divide by $p!$. Carrying out the functional integration then gives

$$\rho_t^{(p)}(x) = \frac{1}{(p-1)!} \prod_{l=2}^{p} \int_{-\infty}^{\infty} dx_l \, |Q_t^{(p)}(x, x_2, ..., x_p)| / P_t^{(p)} \tag{4.2}$$

which has the normalization

$$\int_{-\infty}^{\infty} \rho_t^{(p)}(x) \, dx = p \tag{4.3}$$

We will restrict our attention to the case $t \gg p$. The exponentials in the product in (1.2) can then be expanded to first order and we obtain, after the change of variable $X_k = (x_k - ap/2)/(Dt)^{1/2}$,

$$\rho_t^{(p)}(x) \sim \frac{p}{(Dt)^{1/2}} \frac{\prod_{l=2}^{p} \int_{-\infty}^{\infty} dX_l \, e^{-(X_l)^2/2} \prod_{1 \leqslant j < k \leqslant p} |X_k - X_j|}{\prod_{l=1}^{p} \int_{-\infty}^{\infty} dX_l \, e^{-(X_l)^2/2} \prod_{1 \leqslant j < k \leqslant p} |X_k - X_j|} \tag{4.4}$$

where in the numerator

$$X_1 = (x - ap/2)/(Dt)^{1/2}$$

The integrals in (4.3) have occurred before in the statistical theory of energy levels (i.e., theory of random matrices),[3] and for p even have been exactly evaluated to give

$$\rho_t^{(p)}(x) \sim \frac{1}{(Dt)^{1/2}} \left[\sum_{j=0}^{p-1} \phi_j^2 \left(\frac{x-ap/2}{(Dt)^{1/2}} \right) + \left(\frac{p}{2} \right)^{1/2} \phi_{p-1} \left(\frac{x-ap/2}{(Dt)^{1/2}} \right) \right.$$
$$\left. \times \int_0^{x/(Dt)^{1/2}} dy\, \phi_p(y) \right] \tag{4.5}$$

where

$$\phi_j(u) = (2^j j!\, \pi^{1/2})^{-1/2} e^{-u^2/2} H_j(u) \tag{4.6}$$

$H_j(u)$ denoting the Hermite polynomial of order j. For p large (but still $t \gg p$), this tends to the "semicircle law"[3]

$$\rho_t^{(p)}(x) \sim \begin{cases} (2p/Dt)^{1/2} \left[1 - (x-ap/2)^2/(2p\,Dt)\right]^{1/2}, & |(x-ap/2)| < (2p\,Dt)^{1/2} \\ 0, & |(x-ap/2)| > (2p\,Dt)^{1/2} \end{cases} \tag{4.7}$$

Note that (4.7) is symmetrical about the point $x = ap/2$, the midpoint of the initial configuration of the walkers.

The result (4.7) is nothing but the background density implied by the Coulomb gas interpretation of the integrand in (4.3), and thus could have been derived as the solution of the integral equation

$$\int_{-p/2\eta}^{p/2\eta} \rho_t^{(p)}(y) \log \left| y - \frac{x-ap/2}{(Dt)^{1/2}} \right| dy = \frac{(x-ap/2)^2}{2Dt} + \text{const} \tag{4.8}$$

[recall the discussion between (2.13) and (2.16)].

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